

Propagation of chaos for the Boltzmann equation

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Layout

1. The Boltzmann equation

2. Kac's program

3. Propagating chaos

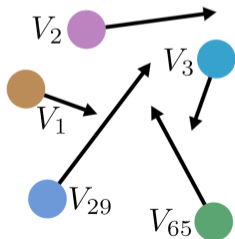
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Picture a system of $N \gg 1$ interacting 'particles' (e.g. molecules, cells, opinions, oscillators,...). At which scale can you consider the system ?

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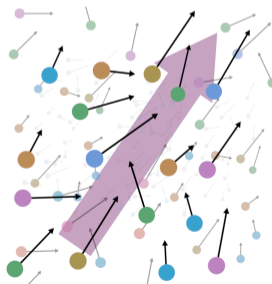
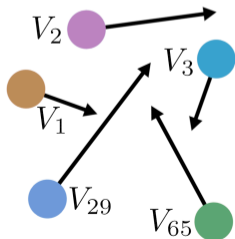
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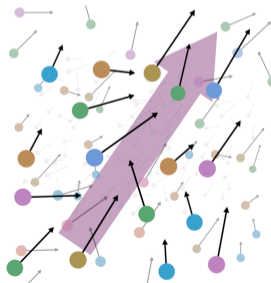
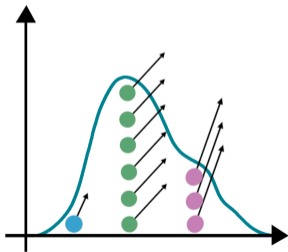
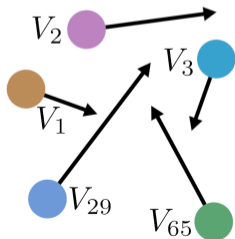
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- **Microscopic:** N coupled differential equations
- **Mesoscopic:** Statistical behaviour of a 'typical' particle
- **Macroscopic:** Mean/thermodynamical quantities



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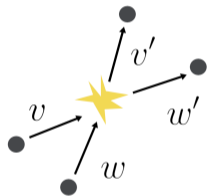
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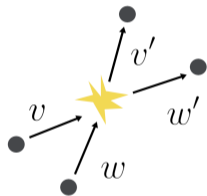
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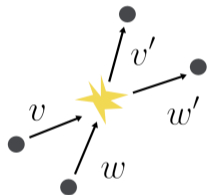
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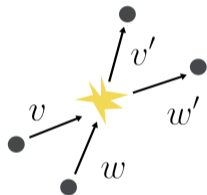
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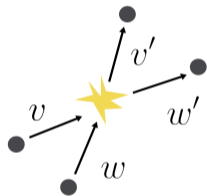
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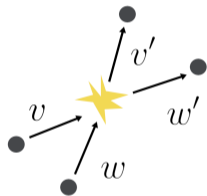


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where f_2 : distribution of pairs of particles,
 B : collision rate.

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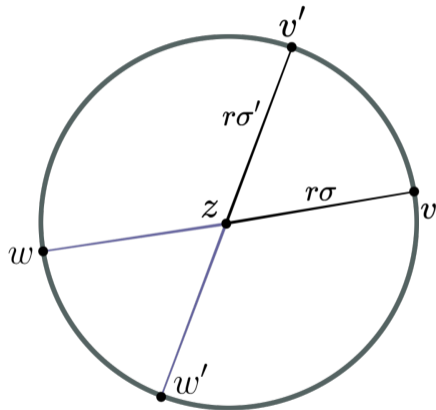
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\rightsquigarrow But what are w, v', w', B, f_2 ???

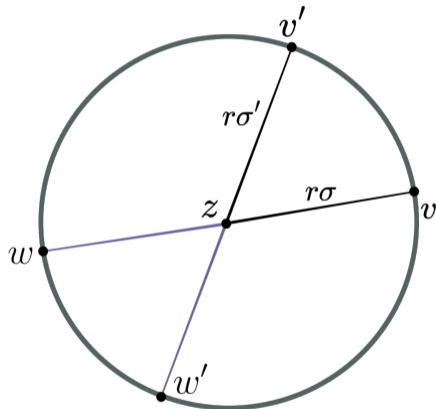
Geometry of collisions



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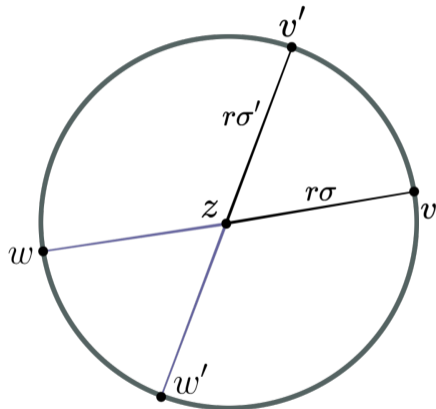
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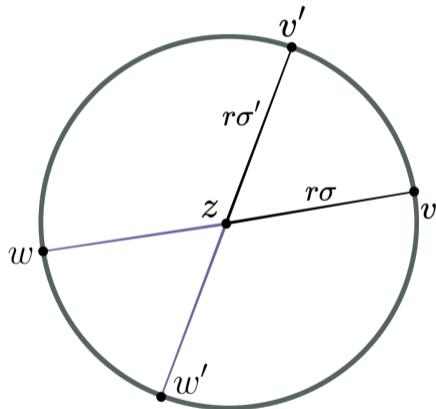
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$$B_{v',w' \rightarrow v,w} = B_{v,w \rightarrow v',w'} = B(r, \sigma \cdot \sigma')$$

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Finally, the Boltzmann equation [Maxwell 1867, Boltzmann 1872]:

$$\partial_t f_t + v \cdot \nabla_x f_t = \iint_{\mathbb{S}^2 \times \mathbb{R}^3} (f_t(v') f_t(w') - f_t(v) f_t(w)) B(r, \sigma' \cdot \sigma) d\sigma' dw$$

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Question (Kac, 1956)

Collisions perpetually **correlate** particles yet **molecular chaos** postulates a form of perpetual **independence**. Can one prove that **molecular chaos** emerges from the microscopic level in the large number limit ?

Kac's program

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Compare to **Boltzmann**:

$$\partial_t f_t = \int Q_{vw}(f_t(v)f_t(w))dw$$

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Propagation of Chaos / Kac's Program

Assume **initial chaos** (independence) $F_0^N = f_0^{\otimes N}$ for some probability distribution f_0 .
Show that for all times $t > 0$, all j ,

$$F_t^{N:j} \xrightarrow{N \rightarrow +\infty} f_t^{\otimes j}$$

where f_t solves the Boltzmann equation.

Propagating chaos

Proving propagation of chaos

Difficulties:

- Infinite number of variables as $N \rightarrow +\infty$
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Approaches:

- **Analytic:** Infinite hierarchy on the $F^j = \lim F^{N:j}$ [BBGKY hierarchy], try to show that $F^j = f^{\otimes j}$ is the only solution. **Harder than uniqueness for Boltzmann.**
- **Probabilistic:** [Sznitman 1991] Work on the particle system itself (rather than its law), rephrase chaos on a common space for all N, j .

Proving propagation of chaos - Sznitman's approach

Key idea: The map

$$\begin{aligned} \mu : \quad \mathbb{R}^{3N} &\longrightarrow \mathcal{P}(\mathbb{R}^3) \\ (v_1, \dots, v_N) &\longmapsto \frac{1}{N} \sum_j \delta_{v_j} \end{aligned}$$

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Let $\mu^N(t) := \mu(V_1^N(t), V_2^N(t), \dots, V_N^N(t))$ be the **random** empirical measure of the system of N particles. **Almost no information is lost.**

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$$\forall j, F_t^{N:j} \xrightarrow{\mathcal{P}(\mathbb{R}^{3j})} F_t^{\infty:j} \iff \mathcal{L}(\mu_t^N) \xrightarrow{\mathcal{P}(\mathcal{P}(\mathbb{R}^3))} F_t^\infty$$

where

$$F_t^{\infty:j} = \int_{\mathbb{R}^3} \rho^{\otimes j} F_t^\infty(d\rho).$$

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\rightsquigarrow Trade-off: $(F^{N:j})_j$ replaced by the 'self-sufficient' μ^N , but which is less regular, less explicit, and random.

The Key Lemma

$$\mathcal{L}(\mu_t^N) =: \hat{F}_t^N \xrightarrow{\mathcal{P}(\mathcal{P}(\mathbb{R}^3))} F_t^\infty$$

\Downarrow

testing the convergence with $\rho \mapsto \int \varphi d\rho^{\otimes j}$

$$\forall j, \int \rho^{\otimes j} \hat{F}_t^N(d\rho) = \hat{F}_t^{N:j} \xrightarrow{\mathcal{P}(\mathbb{R}^{3j})} F_t^{\infty:j}$$

\Downarrow

because $\hat{F}_t^{N:j}$ and $F_t^{N:j}$ are explicitly close

$$\forall j, F_t^{N:j} \xrightarrow{\mathcal{P}(\mathbb{R}^{3j})} F_t^{\infty:j}$$

Applying the key lemma

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Original goal

$$F_t^{N:j} \xrightarrow{\mathcal{P}(\mathbb{R}^{3j})} f_t^{\otimes j} \quad \text{where } f_t \text{ solves the Boltzmann equation.}$$

Applying the key lemma

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$$F_t^{\infty:j} = \int_{\mathbb{R}^3} \rho^{\otimes j} F_t^\infty(d\rho).$$

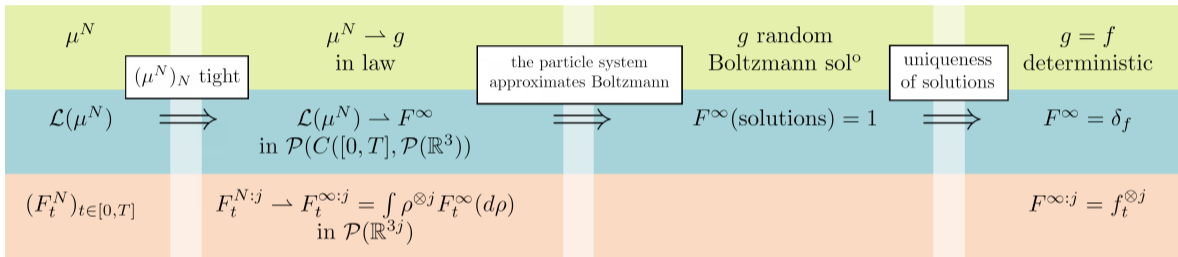
Original goal

$$F_t^{N:j} \xrightarrow{\mathcal{P}(\mathbb{R}^{3j})} f_t^{\otimes j} \quad \text{where } f_t \text{ solves the Boltzmann equation.}$$

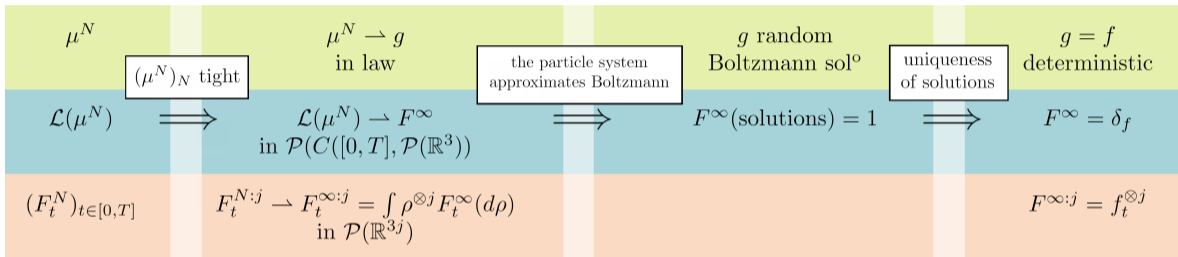
New goal

$$\mathcal{L}(\mu_t^N) \xrightarrow{\mathcal{P}(\mathcal{P}(\mathbb{R}^3))} \delta_{f_t} \quad \text{i.e.} \quad \mu_t^N \xrightarrow{\mathcal{L}} f_t$$

The proof strategy



The proof strategy



Tightness - Consistency - Uniqueness method.

Every step requires **increasingly good estimates** for **increasingly singular** B .

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What do we estimate ? In which 'norm' ?

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- We need quantities that behave nicely with respect to dimension and marginals.
 L^p / Sobolev norms do not.

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Theorem (Imbert-Silvestre-Villani 2024)

The Fisher Information

$$I(f) = \int_{\mathbb{R}^3} f |\nabla \log f|^2 dv$$

is non-increasing along the Boltzmann equation (for virtually all kernels B).

- N -particle counterpart : $I(F^N) = \frac{1}{N} \int_{\mathbb{R}^3} F^N |\nabla \log F^N|^2 dv$ is non-increasing along the Master Equation.

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(\rightsquigarrow same properties as the entropy $\int f \log f$ but higher order control!)

Consequences of Fisher bounds

$$I(f_0) = I(F_0^N) \geq I(F_t^N) \geq I(F_t^{N:2})$$

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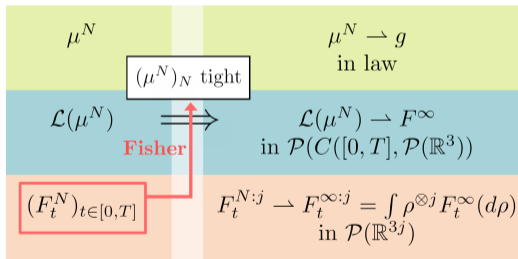
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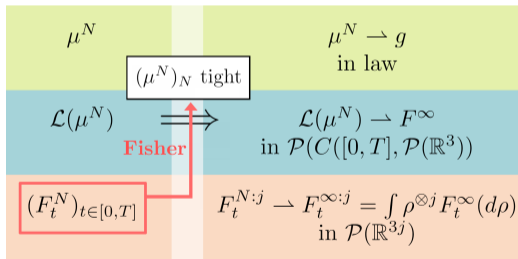


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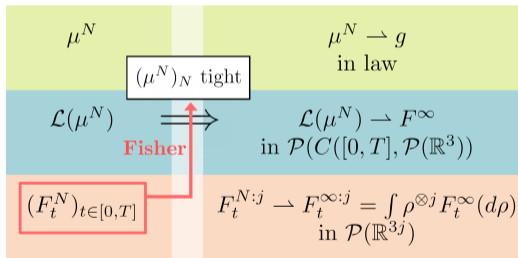


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Next: Which estimate can we obtain on the limits ?

Passing to the limit

$$I(F_t^{\infty:j}) \leq \liminf_N I(F_t^{N:j}) \leq I(f_0).$$

What does this mean on $F_t^{\infty} = \mathcal{L}(g_t)$?

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$$I(F_t^{\infty:j}) = I\left(\int_{\mathcal{P}(\mathbb{R}^3)} \rho^{\otimes j} F_t^\infty(d\rho)\right) \leq \int_{\mathcal{P}(\mathbb{R}^3)} I(\rho^{\otimes j}) F_t^\infty(d\rho) = \int_{\mathcal{P}(\mathbb{R}^3)} I(\rho) F_t^\infty(d\rho)$$

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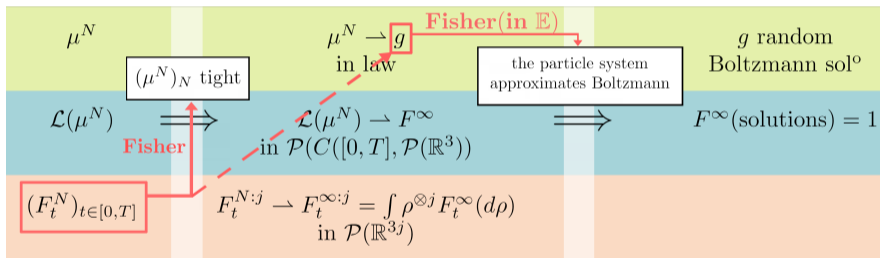
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At best we recover $\int_0^T I(g_t) dt < +\infty$ a.s., which implies $\boxed{g \in L_t^1 L_v^3}$ a.s.

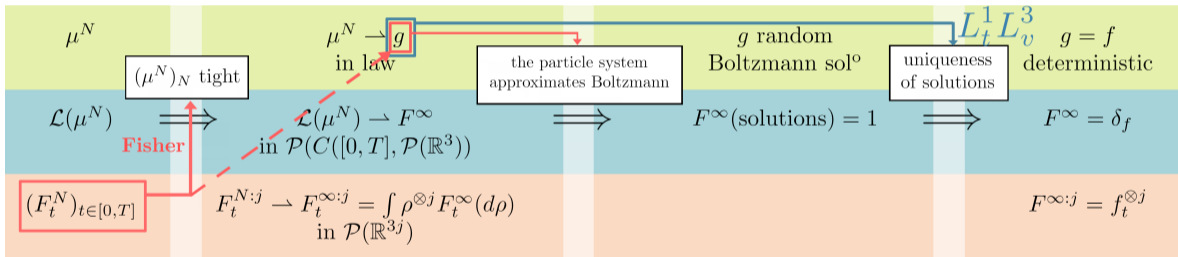
Consequences of limit Fisher bounds

The limit Fisher bounds are enough for consistency for all kernels B .



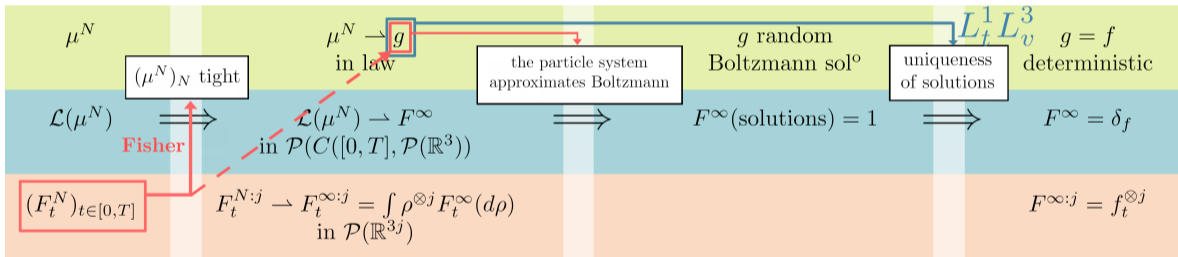
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\rightsquigarrow Covers $\gamma \in (-2, 1)$ but the physically relevant range is $\gamma \in (-3, 1)$.

Final words

Tackling $\gamma \in (-3, -2]$:

One needs a stronger functional than Fisher information.

$$\frac{d}{dt}I(F_t^N) \leq 0 \quad \rightsquigarrow \quad \frac{d}{dt}I(F_t^N) + D(F_t^N) \leq 0$$

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THANK YOU FOR YOUR ATTENTION!

Closeness of $F_t^{N:j}$ and $\widehat{F}_t^{N:j}$ (Grünbaum's Lemma)

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